APPLIED GAME THEORY

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An Introductory Course to Game Theory
Part I: Methodology

INTRODUCTION TO STRATEGIC THINKING

1. **What is common in fare-dodging, the cold war, corruption, and Edward Snowden's case?**

Snowden's Case:

Individual interest	⟷	Collective Interest
(his moral conviction)		(national security)

Illustrative example

- **Fare-dodging:**
  - ✓ **C**ooperation: To buy the ticket
  - ✓ **D**efection: To be a free-rider
  - ✓ Individual interest (D) ←→ Common interest (C)
  - ✓ For a long-run, individual interest is the same as common interest.

Fair Passengers

Preference Profile: DC CC DD CD

2 1 -1 -2

Fare dodger

Exploited passenger(s)

Corruptness
&
CRACK

Social Interest: DD → CC

Game Theory and Social Life (#2)

2. ... and when GT doesn't work properly

Journal of Family Psychology: **Why attractive women match with/marry plain men?**

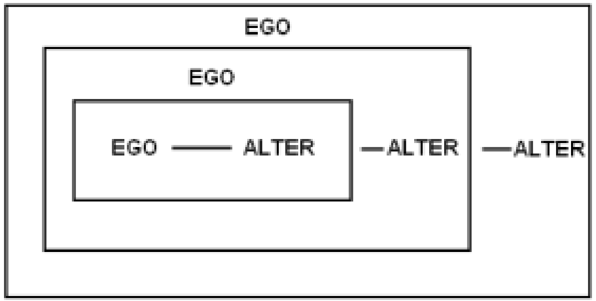
How to answer the question in GT? → Men in general may follow two strategies:

1. **"Daddy" strategy**: they prefer long relations, is ready to bring up her children.
2. **"Rogue" strategy**: they have many short relations, tend to missteps, and so though women appreciate their look but uncertain about their fidelity and ability to help at home.

The model says something (e.g. it predicts that the fidelity depend on the ratio of men and woman in the considered community), but there is a prejudice in the putative strategies of men: are there only two strategies? All men can be categorized in this way? All hot guys are surely untrue? etc.

I. RATIONAL CHOICE PARADIGM

I.1. Theory of Actions and Game Theory



↦ **EGO-ALTER nested relationships** : they are evolved by the articulation of individual (EGO) interests.

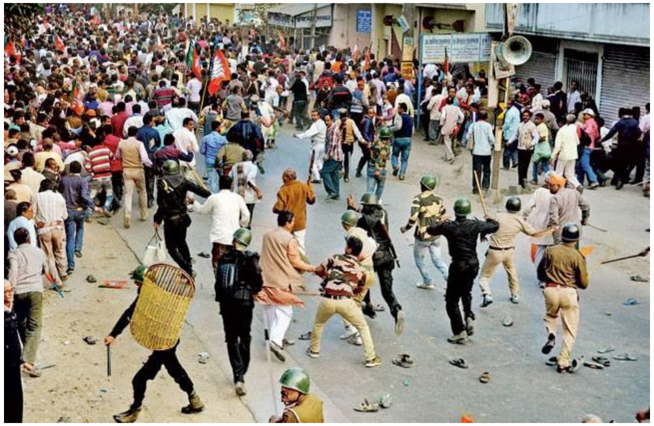
GT is a mathematical theory, its scope is Strategic Interactions. This means two endeavours:

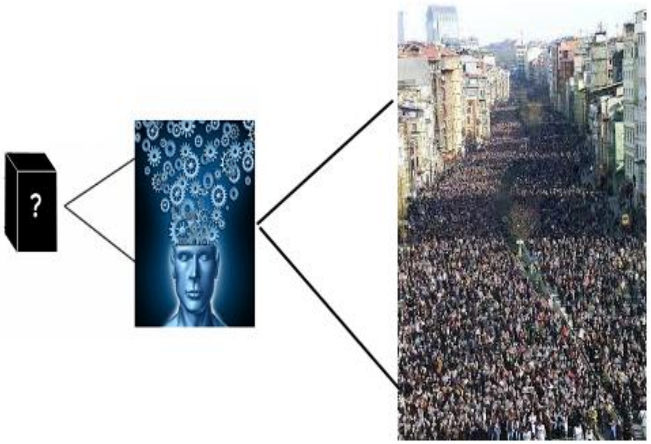
1. **Ontological Requirement:**
   - Society is supposed to be an ensemble of people having opposed, mixed or similar ambitions (interests) which govern them.

Society = Individuals + Their Connections (EGO-ALTER relationships)

2. How do players think strategically?
   - People work out strategies based on surveying their own opportunities and building up preferences over the set of alternatives.
   - Actions (both individual and collective) based on efficacy (hypothesis of rationality) even if it is not consciously and precisely (conception of bounded rationality).

I.2. Theory of Actions and Game Theory



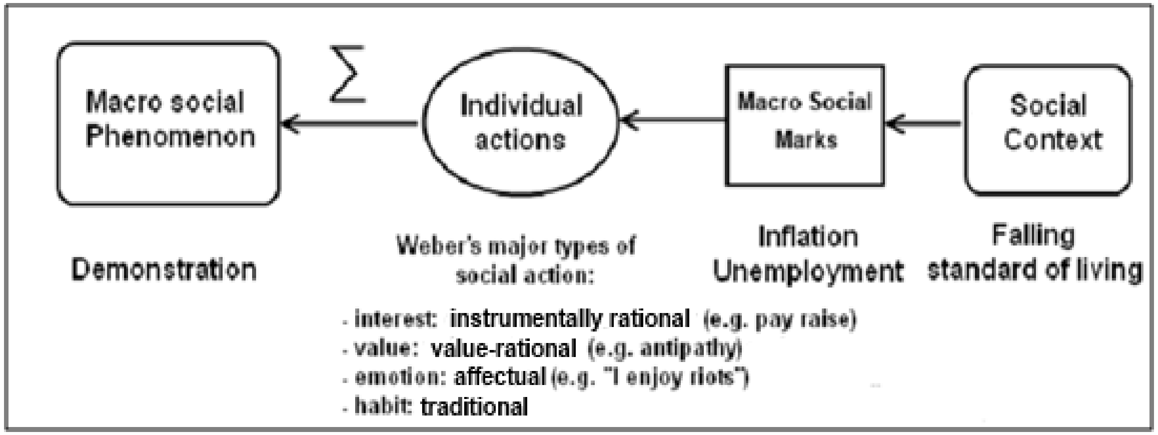


Epistemic Requirement

- The study always begins with individual actions;
- Any social phenomenon is traced back to individual actions.

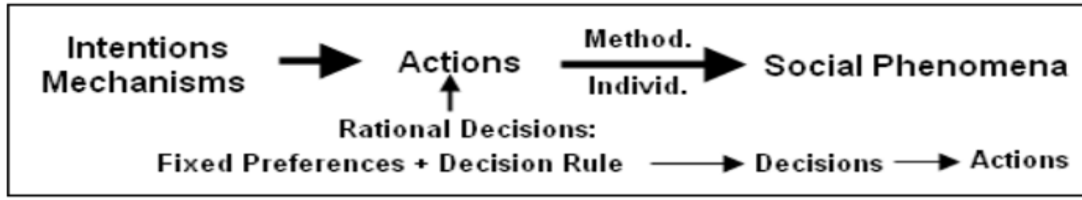
Two types of collective actions:

1. **Contingent** (e.g. incidents) – ***Weberian Tradition #1***: any macro social phenomenon is conceived as a cumulative effect of individual actions with different motivations.



I.3. Theory of Actions and Game Theory (#3)

2. **Regular (repeated) actions – *Weberian Tradition #2***: how do we get same reply to a set of social actions.



To conceive regular actions, we should explore social mechanisms by which social actions are realized.

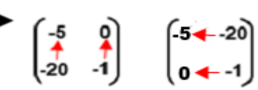
The sitch of collective actions has specific structural-logical framework, and we can characterize it by GT. We have some Metaphors:

- Zero-sum game: "Matching Pennies" Game
- Prisoner's Dilemma
- "Invisible Hand" Game

****

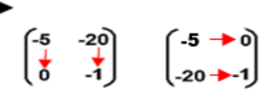
	D	C
D	-5, -5	0, -20
C	-20, 0	-1,- 1

DC CC DD CD



	D	C
D	-5, -5	-20, 0
C	0, -20	1, 1

CD CC DD DC



Player A \ Player B	h	t
H	1, -1	-1, 1
T	-1, 1	1, -1

Each of two players puts down a coin on tht table without letting the other player to see it.

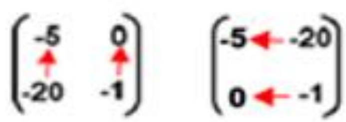
Player A is the winner if the coins match, i.e.,both coins show heads (Hh) or both show tails (Tt).

Player B is the winner if the coins do not match (Ht or Th).

Invisable Hand Game:

	D	C
D	-5, -5	0, -20
C	-20, 0	-1,- 1

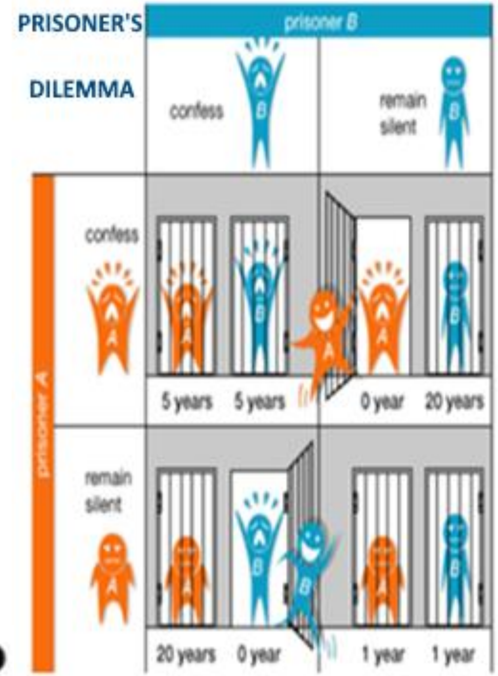
DC CC **DD** CD



	D	C
D	-5, -5	-20, 0
C	0, -20	1, 1

CD **CC** DD DC

Adam Smith's story



COMPANION	ME	
CONFESS	CONFESS	-5
	DENY	-20
DENY	CONFESS	0
	DENY	-1

PREFERENCE: DC CC DD CD

PAYOFF: 0 -1 -5 -20

EQUILIBRIUM

Two prisoners are suspected of taking part in a serious crime and shut up in separate jails. The punishment depends on whether or not they confess. If both confess, they will be sentenced to five years. If neither confesses, both will get a sentence to one year on account of a lesser guilt. If one confesses and the other does not, the former will be free, while the other will receive a severe sentence of twenty years. What should they do?

I.5. Rational Choice Theory as Research Program

◈ The Tradition and the point of Rational Choice Theory

The study always begin with individual actions. We have two types of collective actions to consider:

Weberian Tradition #1 – Contingent actions (e.g. incidents): any social phenomenon is conceived as a cumulative effect of individual actions with different personal motives.

Weberian Tradition #2 – Regular (or repeated) actions: To conceive regular actions, we should explore and interpret 1) the individual behaviours that is ”adequate with respect to sense”; and 2) social mechanisms by which social actions are realised in social practice.

Rationality:

- Human behaviour is not random;
- Human actions usually do not happen unpredictable, or self-destructive manner;

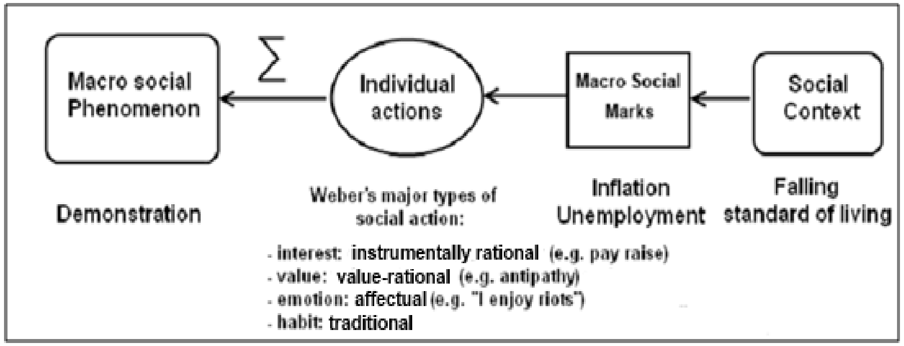
→ We can make the ”instrumentalization of actions”

→ Actions are embedded in strategic interactions

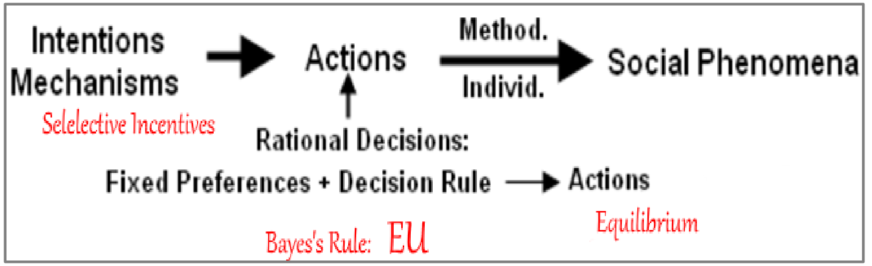
→ Rationality is always bounded

→ Individual rationality can be extended to collective rationality.

Weberian Tradition #1



Weberian Tradition #2



I.6. What is Instrumentalization of Actions?

- **Rationality as self-interest:**

A generic can of beer (”Garage Project”): all beer is beer, but not all beers are the same. Right?

”Instrumentalization of actions”: self-interest is a generic concept. And it is as the same as a generic can of beer.

- ✓That is to say, all behaviour is self-interested, but not all self-interested behaviour (among individual actors) is the same.
- ✓ Specific self-interests can and usually are quite distinct, and these distinctions are often very important.
- ✓ **Examples:**

The interest of a **politician** is to win or **hold on to political office**.

A **student** wants **good grades** (though not always)

A **business person** wants to stay in business and **maximize profit**

I.7. Rational Choice

- Strategic interaction emphasizes that many decisions are complicated by the existence of other actors.
- The scale illustrates the "weighing of costs and benefits";
- Poker represents the dynamics of strategic interaction.

1	2	3	4		Playsers
10	0	0	0	«–	All for himself
2	2	2	2	«–	Cooperation
U_1	U_2	U_3	U_4		

Bentham: $W = U_1 + U_2 + U_3 + U_4$ $W_D = 10$ > $W_C = 8$

Nash: $W = U_1 \times U_2 \times U_3 \times U_4$ $W_D = 0$ < $W_C = 16$

Hollywood Movie:
The Beautiful Mind. The Bar Scene.

Defection: all guys go for the blonde

Cooperation: no one goes for the blonde, instead the four boys pair up with the other girls

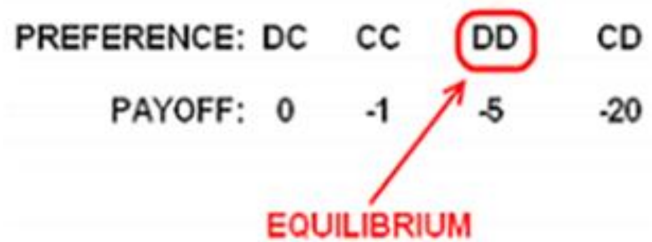
I.8. What is Rational Choice Theory? (#1)

RCH is a generalzation of GT, and there is a close connection between them:

Self-interest

Decision Sitch:
- Decision maker
- Alternatives
- Preferences

Game Sitch:
- Players
- Stratagies
- Payoffs



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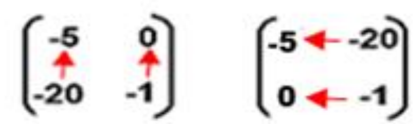
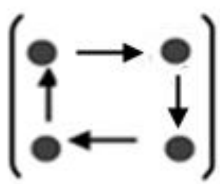
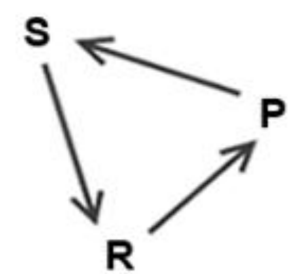
DC CC DD CD

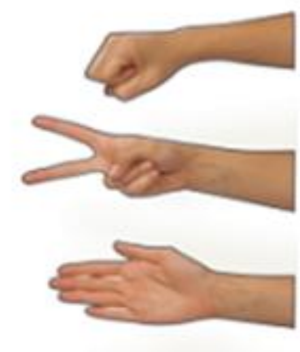


No Equilibrium:



Preference cycle:





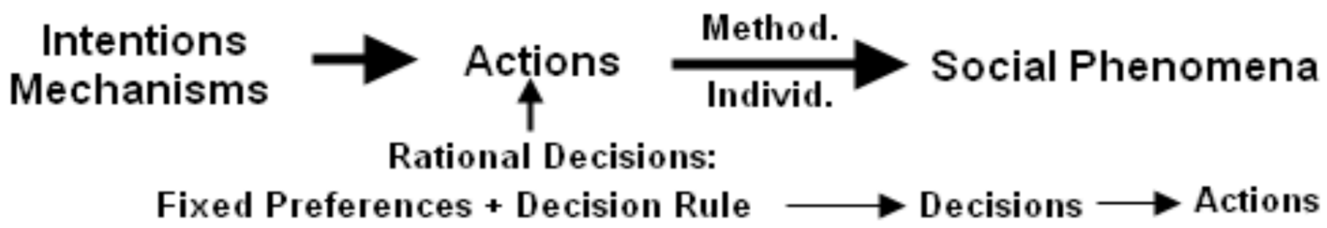
Rock-Paper-Scissors

#1/#2	Paper	Rock	Scissors
Paper	0	1	-1
Rock	-1	0	1
Scissors	1	-1	0

I.9. What is Rational Choice Theory? (#2)

The analysing levels of Rational Choice Theory

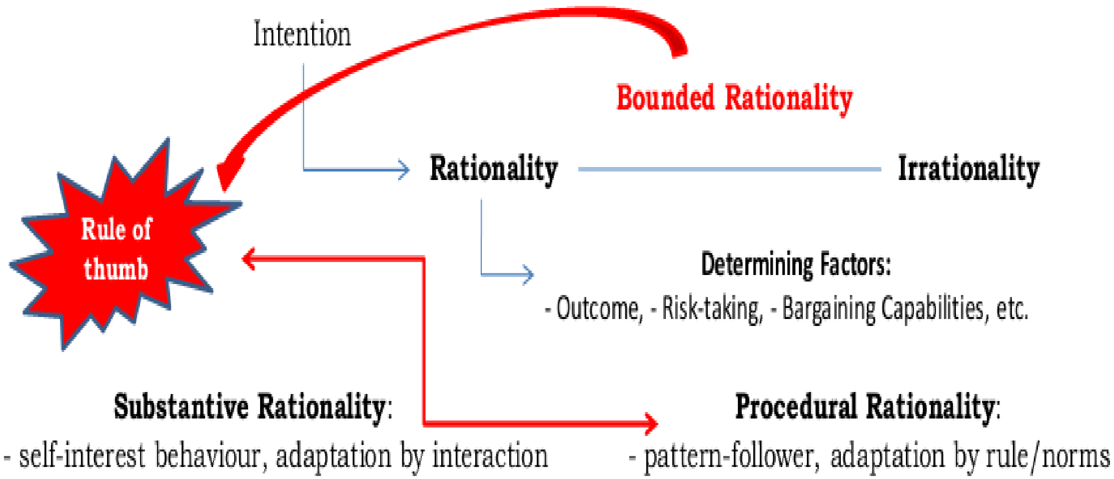
1. *Subintentional Level* : to explain the formation and the alteration of preferences by mechanisms.
2. *Intentional Level* :



3. *Interaction Level* (including Game Theory): The goal is to explain interactions with intentions. Based on the putative and real preferences of actors, and supposing they do not want to come off badly, we make an attempt to describe and to grasp the logic of the situation, and to predict the expected outcome.
4. *Superintentional Level* : Consequences going beyond the intentions of the actors.

ON FOCUS: RATIONAL ITY AND MECHANISMS

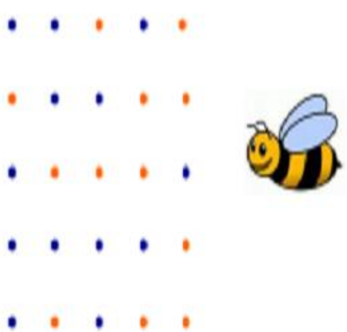
The Sketch of the Idea:



The Main Exponents: Herbert Simon, Daniel Kahneman, Thomas Schelling, Jon Elster

- ✓ **Rationality** is an assumption that self-interest is the basis for most of what we do, which can be stimulated by some factors.
- ✓ However, since the world is complex, and in strategic interactions there are a lot of ”unknowns”, less than optimal decisions are part and parcel of making rational choices. That what we call as **bounded rationality**.

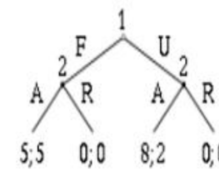
Case Study #1:



Wacky bee?

Ingenious bee?

Ultimatum Game



The persons interact to decide how to divide a sum of money that is given to them. The first give a proposal about the distribution, the second decides either accepts it or not. If the second rejects, neither player receives anything. Experimental result: **all offers of less than 20% are often rejected.**

Wacky or Proud Humans?

I.11. Rationality (#2)

◈ Case Study #2:

Oral Sex has an increasing trend among teens from 1980s. Explanation?

#1: Fashion, too much porn via media ← **procedural rationality**

#2: Sexual infections (AIDS, Hepatitis), stricter abortion laws in many states ← **substatntive rationality**

Determining Factors of Rationality:
- Outcome;
- Risk-taking

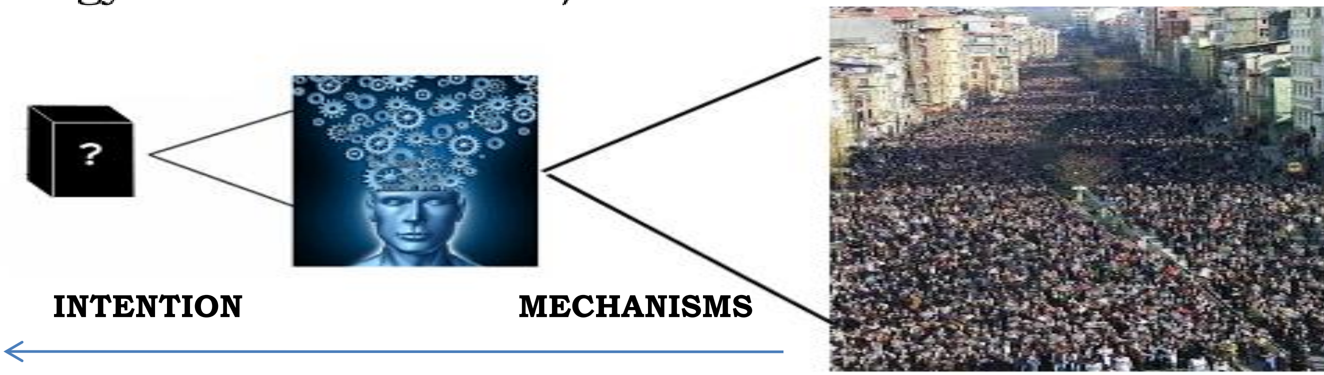
◈ Case Study #3:

McNamara
- Human faculty is in fact bounded
- The origin of bounds is more than social enviroment
- **Post-positivist perspective:** Bounded rationality is based on the complexity of choice

Foucault
- The whole business of rationality is dubious
- Rationality is just one of our cultural patterns
- **Postmodern perspective:** Bounded rationality solely rested on procedural rationality

I.12. Theory of Actions: Weber's Interpretive Sociology

- **What is "social action"?** (M. Weber: *Economy and Society*, Ch. 1: "The definition of Sociology and Social Action")



Causally adequate interpretation of a sequence of events

"We shall speak of 'action' insofar as the acting individual attaches a subjective meaning to his behaviour [...] Action is 'social' insofar as its subjective meaning takes account of the behaviour of others and is thereby oriented in its course."

"A motive is a complex of subjective meaning which seems to the actor himself or to the observer an adequate ground for the conduct in question. The interpretation of a coherent course of conduct is "adequate with respect to sense" (*Sinnadäquanz*) to the extent that the relationship between its composite parts is confirmed by us as a typical context of sense (*Sinnzusammenhang*) according to cross sectional usages of thought and feeling."

There are three keystones in exploring social actions:
1. **Intention:** individual action is "adequate with respect to sense".
2. **Social Mechanisms**: to roadmap relevant circumstances fitted in the given "social context".
3. **Such kind of actions are oriented to others**.

So, all in all, if a conduct is to qualify as a comprehensible social action in a social context it must be given rational action-schemes by which the actor is "adequate with respect to sense" in a given "social context".

I.13. Social Mechanisms: Case study

- **Selective Incentives:**
  1. One mechanism to make people participate in collective action.
     - ✓ They are ”selective” because they are individually and not jointly supplied.
  2. Selective incentives can be: 1) moral or material; 2) positive or negative.
     - ✓ **Example:**
       - Material, positive selective incentives: gifts, perks for members such as insurance, clubhouses, and discounts in shops, hotel or car rental.
       - Negative ones can consist of fees, fines, or taxes.
       - Moral incentives: prestige, awards, access to social netwoks.

- **Definition:** Selective incentives are private goods provided conditionally to the participants in collective action.
  - **Public goods:** Goods that are *inclusive*, i.e., jointly consumed; collectively desirable, and are *jointly supplied* to their potential users. *Must share in potential utilities and damages.*
  - ✓ **Example:** sea, roads, protection and security, etc.

We make different game situations to consider the intentions of actors and the interdependencies of their choices:

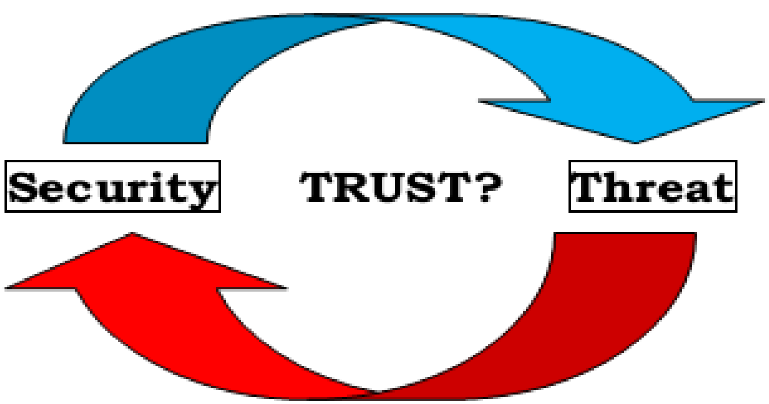
In IR Theory:

- **The Cold War** → ***Prisoner's Dilemma*** / ***Security Dilemma*** / ***Perceptual Dilemma***

> **As states acquire capabilities to make themselves secure, they make others more insecure – leads to a cycle of arms races and growing insecurity.**

Our viewpoint (EGO): ***Security Dilemma***

The Others (ALTER): ***Prisoner's Dilemma***



I.15. Mechanisms in GT (#2)

Mechanism engineering in GT depends on the concept of Rationality:

Traditional (Neumann-Nash) Game Theory
Substantive Rationality

Evolutionary Game Theory
Procedural Rationality

Population:
An ensemble of players acting similarly due to playing the same strategy

Applications:

#1: Who cares of offsprings (only the male, only the female, both of them, or none of them)?

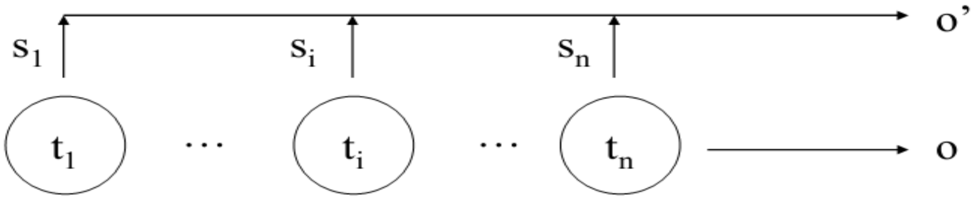
#2: Why is it worth playing rituals in evolutionary sense?

#3: The Paradox of Rational Voter (Downs paradox)?
It is non-rational at all for a rational, self-interested voter to vote, because the costs of voting will normally exceed the expected benefits. → Renormalized Rationality or Superrationality

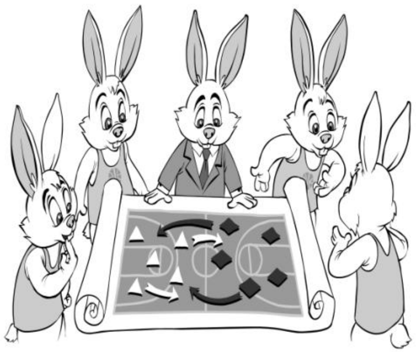
I.16. Warning!

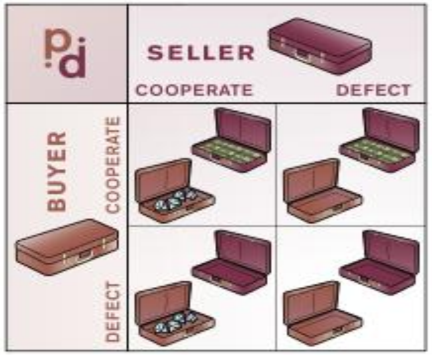
- **Mechanism engineering ≠ Mechanism designing**

  ↑ **General, methodological concept** ↑ **Game-theoretical concept**



- **Mechanism Design = Inverse Standard Game Theory**
  - **Standard GT:** we are looking for optimal actions (a set of strategies) leading to equilibrium.
  - **Mechanism Design:** we are designing a game to a set of strategies (a certain "type" of strategies) to lead a desirable outcome (equilibrium). That is to say,
    - ✓ a game designer defining the structure of the game;
    - ✓ the game designer is interested in specific outcomes and make attempts to influence players' behavior to achieve these outcomes.
    - ✓ These games belong to the realm of incomplete information games.





I.17. Types of Players: Nature and Omnipotent Player

- In Game Theory we analyse strategic interaction games where there are strategic players. We specify them as a set of players between two extraordinary types of players, Nature and Omnipotent Player.

1) **Nature as player (Parametric Decision Situation)**
What is Nature from game theoretical viewpoint? It is a non-rational player with no interest whose goal is not to play systematic game with her opponents.

NATURE:

Both DETERMINISER AND RANDOMISER

2) **Omnipotent Player**

Newcomb Paradox. A world-power Being who is infallible puts money in two boxes, one transparent (labeled A) and the other opaque (labeled B). A player is permitted to take the contents of both boxes, or just the opaque box B. Box A contains a visible one thousand dollar. The contents of box B, however, are determined as follows: at some point before the start of the game, the infallible Being makes a prediction as to whether the player of the game will take just box B, or both boxes. If the Being predicts that both boxes will be taken, then box B will contain nothing. If the Being predicts that only box B will be taken, then box B will contain one million dollar. How does the player decide?

Under the Condition of DETERMINISM

Newcomb Paradox

Box A	Box B
1000	1.000.000 or Nothing

Under infallibility (pr=1) or well-guessing (pr=0.9), EU(Box B) > EU(Box A&B)

Under the Condition of FREE WILL

Player \ The Being	Do not money in Box B	To put money in Box B
To take Box A&B	1.000	1.001.000
To take Box B	0	1.000.000

I.18. Types of Players: Strategic Players

Holmes-Moriarty Paradox:
The legendary detective Sherlock Holmes is trying to flee from his mortal enemy Professor Moriarty, the criminal genius behind a highly organized and extremly secret criminal force. At Victoria Stat. in London he recognized the professor. Holmes is sure Moriarty is going to hire a one-coach train to follow him. He has to make a decision:
1) He alights at Canterbury, the only sta tion between London and Dover (Strategy C) or
2) he goes on his trip to Dover and then to Calais, France by Ship (Strategy D)

Moriarty \ Holmes	D	C
D	1	−1
C	−1	1

Moriarty \ Holmes	D	C
D	100	0
C	−50	100

There is no equilibrium under pure strategies but it exists under mixed strategies:
Moriarty:(3/5, 2/5) and Holmes: (2/5, 3/5)
D C

1) To survive the situation for holmes is the same as the death of Moriarty.The out-come CC or DD means Moriaty captures Holmes

There is no equilibrium: the situation is very similar to the Matching Pennies Game.

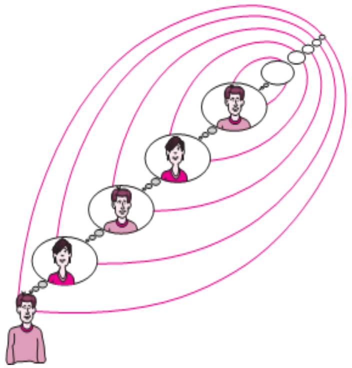
2) Other payoffs: DC is better for Holmes than CD because DC means Holmes leaves for Calais and he can fight Moriarty more effec-tively from the Continent. Thus their pref-ereneces are
Holmes: DC≻CD≻DD=CC,
Moriarty: DD=CC≻DC≻CD.

They worth playing the option with higher probability, since they play the situation only once. So Moriartyplays D and goes to Dover, Holmes play C and get off at Cantenbury

I.19. Collective Representation of Society

- The goal of strategic players is to **overplay** the other(s). It means a spiritual battle between Holmes and Moriarty by claiming "I think/know that...".

  1. Holmes claims his belief: *"I think I should travel to Dover: I am going to step off and to strongly fight Moriarty from the Continent"* .
  2. Holmes claims his belief about the Moriarty's belief: *"I think Moriarty knows what my destination is. That's why I should try to overplay him"* .
  3. The way of thinking might be as follows: *"I think Moriarty knows I am going to travel to Dover, so I think that I have him over: I am going to get off at Canterbury"*. But Moriarty is also able to think it over, etc. *ad infinitum*.



- We can describe the different levels of mind in Holmes–Moriarty situation by using factual knowledge:
  $K_H(E)$ is as a description of primary mind, $K_H K_M(E)$ is as a description of secondary mind, $K_H K_M K_H(E)$ is a description of third mind, etc.
- After the sociologist Émile Durkheim, we should consider the below levels of mind as to be important to create and to keep up a community.
  Primary Mind: That knowledge and beliefs about the world the EGO thinks/ knows about the world. To put it another way, it is all what are in the head of actor(EGO).
  Secondary Mind: That knowledge and beliefs about the world what are in the head of actor(EGO) about the others'(ALTER's) primary mind.

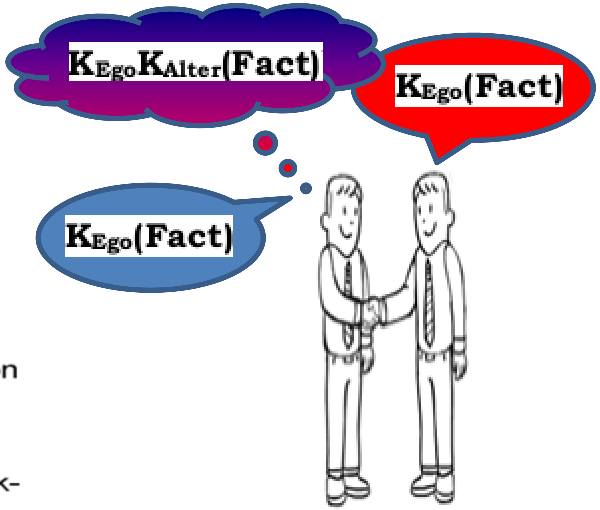
I.20. Collective Representation of Society

Émile Durkheim: De La Division Du Travail Social, 1893
(The Division of Labour in Society)

There is a collection of feeling and beliefs in the common members of society, which consitutes a common, specific system of which has an own life. This is a collective mind.

The presumption of collective mind is that there must be a mutual part of secondary mind. Something about which everybody knows the others know. The absence of this community, "the mind of ours" cannot be evolved.

- **Primary, secondary mind, and common knowledge:**
  - ✓ **Primary:** what I thinks/know about fact: $K_{Ego}(Fact)$
  - ✓ **Secondary:** what I thinks/know about what the other(s) knows/thinks about fact: $K_{Ego}K_{Alter}(Fact)$
  - ✓ **Common knowledge**: Primary mind plus secondary mind plus third mind, and so forth. ← **MODERN INTERPRETATION**



- **Remark:**

  As we can see, mutual knowledge is not enough to solve a coordination problem like Holmes-Moriarty situation. Bad decisions are often based on false secondary knowledge, that is to say misunderstanding the other's position ("I thought you knew that..." situations). To extend "rationality" to a collective level, a higher knowledge than secondary mind (Durkheim's collective mind) is needed. This is common knowledge.

I.21. Illustration of Common Knowledge

There are 3 players (wise men) wearing a hat that is either white or red. Each player can see the hats of the other, but not his own.

Scenario 1: They are asked what colour of his own hat. They cannot answer for sure.

Scenario 2: Someone (the King) announces: "At least one of you wearing a white hat" - this will be a common knowledge among players, and after two questions "what colour of your hat?", the last player can answer for sure: "Yeah, my hat is white!".

Proof:

#1 is asked: he cannot answer

#2 is asked: his knowledge about the situation

- #2 and #3 are not wearing red hats, for #1 could not reply.
- he can see #1 and #3 are wearing white hats.

a) both #1 and #3 wearing white hats

b) #1 and #3 wearing white hats, #2 is wearing red hat

So player #2 will not reply, since he cannot choose between options a) and b).

#3 can answer for sure, since he is able to rule out b) because of the pass of player #2.

II. GAMES AND DECISIONS

II.1. On Games (#1)

- **The main elements of a game** are i) the players, ii) strategies, i.e., a set of feasible actions, iii) each players' payoffs over their preferences.
- **Strategy types**
  - Pure strategy: the strategic actions from all the alternatives available to the player in the game.
  - Mixed strategy: if player randomizes in some manner among his pure strategies.
- Many social issues involve strategic interactions. Games are convenient ways of modelling the strategic interactions among decision makers.

Game Situation : - Players - Strategies - Payoffs ⇄ Decision Situation : - Individuals - Alternatives - Preferences

$$x \succeq_i y \Leftrightarrow v_i(x) \geq v_i(y) \Leftrightarrow Eu_i(x) \geq Eu_i(y)$$

REPRESENTABILITY CRITERION   *DOMINATION PRINCIPLE*

The representability criterion in the prisoner's dilemma:

$$DC \succ CC \succ DD \succ CD \quad \text{Preferences } \succ \textit{ ordinal utility}$$

$$0 > -1 > -5 > -20 \quad \text{Payoffs } > \textit{ cardinal utility}$$

Expected Utility Theory:

	y_1	y_2	
EGO / ALTER	**D**	**C**	
D	v_{11}	v_{12}	← x_1
C	v_{21}	v_{22}	← x_2

- **Payoffs**
  - Zero-sum game: *number*
  - Non-constant-sum game: *vector*

- **Mixed strategies:** *vectors*

For Row Player: (x_1, x_2) For Column Player: $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$x_1 + x_2 = 1$ $y_1 + y_2 = 1$

$$0 \leq x_1,\ x_2,\ y_1,\ y_2 \leq 1$$

- **Payoff Matrix:** *matrix*

$$\mathbf{A}_{2X2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathbf{A}_{2X3} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \longrightarrow$$

In Matrix Notation:

Number: $(\boldsymbol{a_{11}})$

Vector:
- in row matrix: $(\boldsymbol{a_{11}}, \boldsymbol{a_{12}}, \ldots, \boldsymbol{a_{1n}})$
- in coloumn matrix: $\begin{pmatrix} \boldsymbol{a_{11}} \\ \cdots \\ \boldsymbol{a_{1n}} \end{pmatrix}$

II.3. Collective Action Function

- **The Failure of collective actions:**
  - **Free Rider:** Someone who is able to use the public goods without contributing to its provision.
  - If the majority of a community opt for being free rider, they will abstain from participation, and thus collective action will end with no success.
  - This conclusion is true even if the individual longs for the successful result of the collective action.
- **Mancur Olson: The Logic of Collective Actions**

Mancur Olson

The individual logic of deciding whether to participate can be represented by this collective action function:

$$EU = u*p - c$$

where
- EU is the expected utility (reward for an individual for participating in collective action);
- u is the benefit (utility) from accessing the public good;
- p: the probability of the effectiveness of individual action if the individual is ready to participate;
- c: the cost of participation.

Remark: For private goods we can use the same function subject to $p=1$, since for private goods the individual action ("paying the price") always makes a difference to obtain the good.

$$EU = v \times p - c \iff EU = v \times p - c \times (1) \iff EU = v \times p + \underbrace{(-1) \times v \times p}_{-c}$$

$$EU = v_1 \times p_1 + v_2 \times p_2$$

Generalised Bayes's Rule: $EU = \sum v \times p$

$P = X_i \times y_j$ Mixed strategy

In Game Theory: $\mathbf{EU} = \sum v_{ij} \times p$

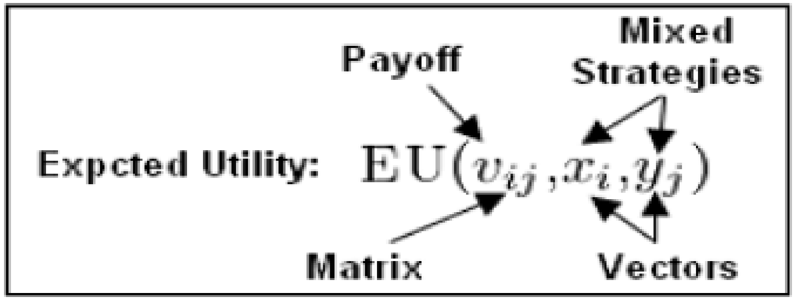
payoffs

$$EU = \sum_{i=1}^{m} \sum_{j=1}^{n} v_{ij} x_i y_j$$

$$P = X_i \times y_j$$

$$\downarrow$$

$$\mathbf{EU} = \sum v_{ij} \times p$$



EUs for Row Player (EGO):

$$EU(D) = x_1 v_{11} y_1 + x_1 v_{12} y_2$$

$$EU(C) = x_2 v_{21} y_1 + x_2 v_{22} y_2$$

EUs for Column Player (Alter):

$$EU(D) = x_1 v_{11} y_1 + x_2 v_{21} y_1$$

$$EU(C) = x_1 v_{12} y_2 + x_2 v_{22} y_2$$

	y_1	y_2	
EGO / ALTER	D	C	
D	v_{11}	v_{12}	$\leftarrow x_1$
C	v_{21}	v_{22}	$\leftarrow x_2$

II.6. ”Matching Pennies” game as Zero-sum game (#1)

	Player A		Player B	
Pure str.:	**H**	**T**	**h**	**t**
Mixed str.:	**¼ or 25%**	**¾ or 75%**	**?**	**?**

$X_1 = \frac{1}{4}$ ← **1** **3** → $X_2 = \frac{3}{4}$

in a 4-round game

In Odds: 1 to 3

In Probability:

$$\frac{\textit{numb of actual strtaegy played}}{\textit{numb of rounds played}}$$

II.7. "Matching Pennies" game as Zero-sum game (#2)

EGO / ALTER	D (h), y_1	C (t), y_2	
H / D	-1 v_{11}	1 v_{12}	x_1
T / C	1 v_{21}	-1 v_{22}	x_2

Player A / Player B	h	t
H	1, -1	-1, 1
T	-1, 1	1, -1

The Goal of Players:

Row Player (Pl. A): $max_{row} min_{col}\ EU = \underline{v}$

Column Player (Pl. B): $min_{col} max_{row}\ EU = \overline{v}$

Minimax Theorem: $\underline{v} \leq \overline{v}$.
If $\underline{v} = \overline{v}$, there exists an equilibrium strategy

The EU of Column Player's strategies:

D → h **y_1 → 1**

$$EU(D) = x_1 v_{11} y_1 + x_2 v_{21} y_1 = v(h|H)\frac{1}{4} + v(h|T)\frac{3}{4} = 1/2$$

($v(h|H) = -1$, $v(h|T) = 1$)

$x_1 = \frac{1}{4}$

$x_2 = \frac{3}{4}$

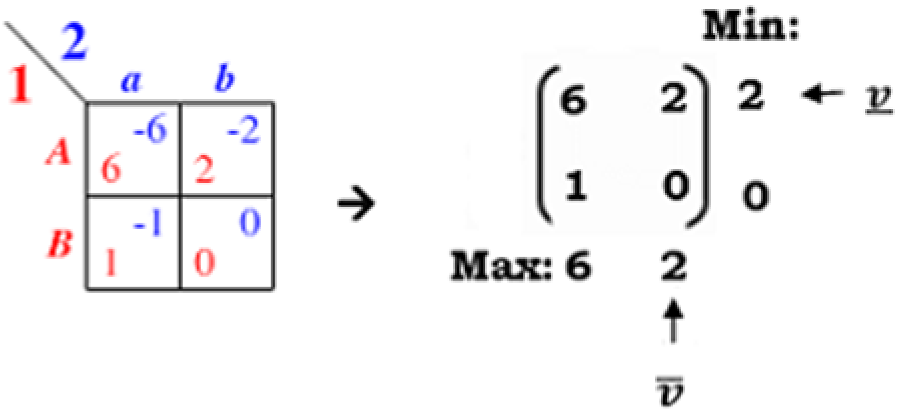
C → t **y_2 → 1**

$$EU(C) = x_1 v_{12} y_2 + x_2 v_{22} y_2 = v(t|H)\frac{1}{4} + v(t|T)\frac{3}{4} = -1/2$$

($v(t|H) = 1$, $v(t|T) = -1$)

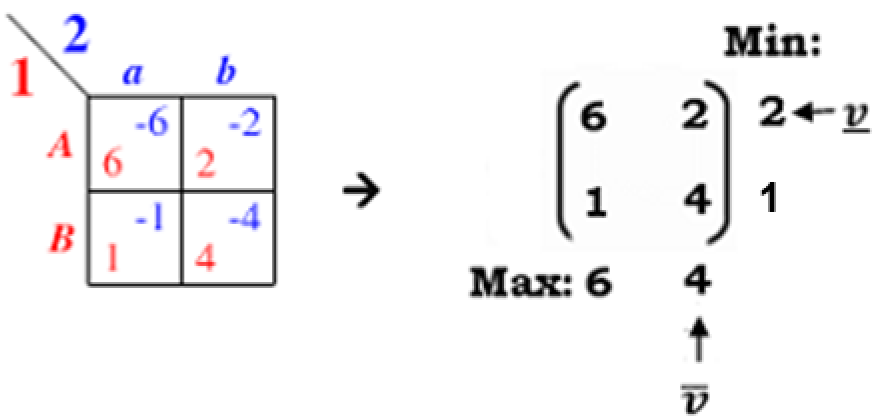
II.8. MiniMax in Practice

Zero-sum Games:



There is eq. in pure strategies:

$\underline{v} = \overline{v} = 2 \rightarrow$ **(A, b) = (2, -2)**

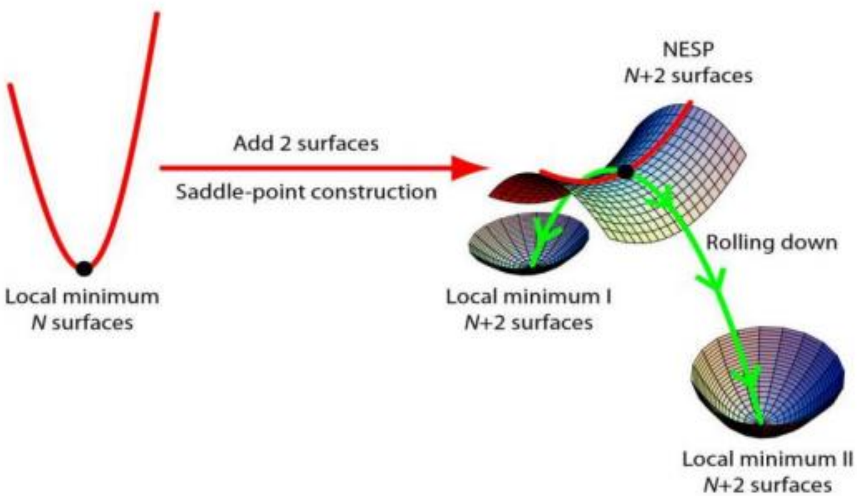


There is no Eq. in pure strategies: $\underline{v} \neq \overline{v}$

II.9. The Interpretation of Equilibrium

- **Informal Definition:** The equilibrium of the game is what the player has to follow if he does not want to come off badly.

- **Equilibrium as Saddle Point**:



		Company B 1	2	3	Row Min
Company A	1	2	4	2	(2)
	2	1	−5	−4	−5
	3	2	6	−2	−2
Column Max		(2)	6	(2)	

Max Min = Min Max

2 = 2

II.10. Interlude: Correlated Equilibrium

- To synchronize the players' strategic choices:

Primitive Randomization → Nash Equilibrium

Find a mediator (or referee) who can perform smart randomization → Correlated Equilibrium

- The players can be 1) self-consistent, 2) "refuser", or 3) "obeyer" → There are several kinds of equilibrium

- Example:

#1 \ #2	L	R
U	5, 1	0, 0
D	4, 4	1, 5

Nash Equilibrium: (U, L), (D, R), [(1/2U, 1/2D), (1/2L, 1/2R)]

Payoffs: (5,1) (1,5) (5/2, 5/2)

What strategy will player choose? Payoffs from coorelated Equilibrium: (10/3, 10/3)

The Mediator (Referee) randomizes over {1, 2, 3}:
if it is
- 1, he tells #1 to play U, and #2 to play L
- 2, he tells #1 to play D, and #2 to play L
- 3, he tells #1 to play D, and #2 to play R

1) if #1 hears U, belives #2 will play L → Play U

2) if #1 hears D, belives #2 will play 1/2L, 1/2R → Play D

3) if #2 hears L, belives #1 will play 1/2U, 1/2D → Play L

4) if #2 hears R, belives #1 will play D → Play R

II.11. Classification of Games

- **Number of players:** how many players are there in the game?

  ”In many-player situations it arises that all the player’s lot depends on the actions of their partners”, and in this case the question is ”how they have to play to get the best result they can [...] hardly can imagine a situation in ordinary life where this problem is not relevant.” (John Von Neumann, 1926)

- **One-shot or repeated games:** how many times do we play a game?

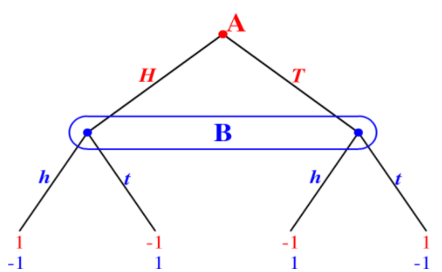
  Advise: If you plan to pursue an aggressive strategy, ask yourself whether you are in a one-shot or a repeated game. And if it is a repeated game, think again.

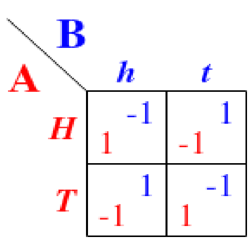
- **Zero-sum game or a non-constant-sum game:**

  ✓ **Zero-sum game:** what is my gain, it is the loss of the others. That is to say, - utility is strongly limited; - actors focus on the obtaining position in the situation; - the other actors are opponents; - allowance failing;

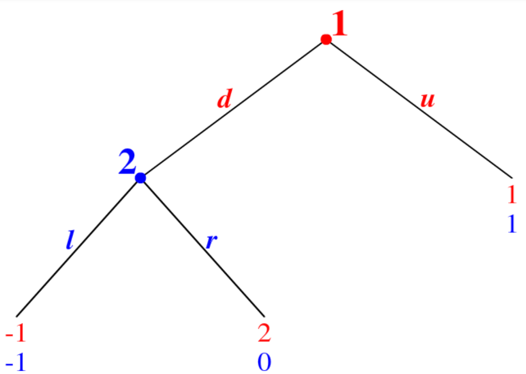
  ✓ **Non-constant-sum game:** there is a chance for all the players to gain (positive-sum game). That is to say, - actors consider mutual interests and they often makes agreement; - they are opponents or often partners with multi-interest;

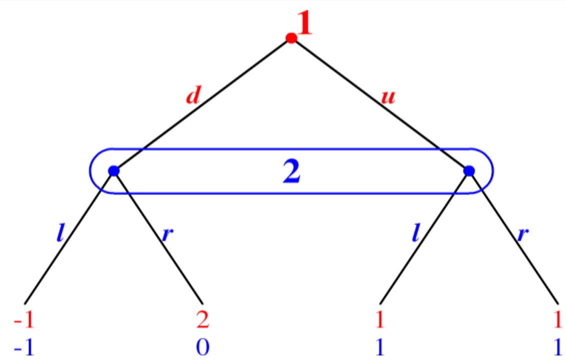
- **Sequential or simultaneous move game**

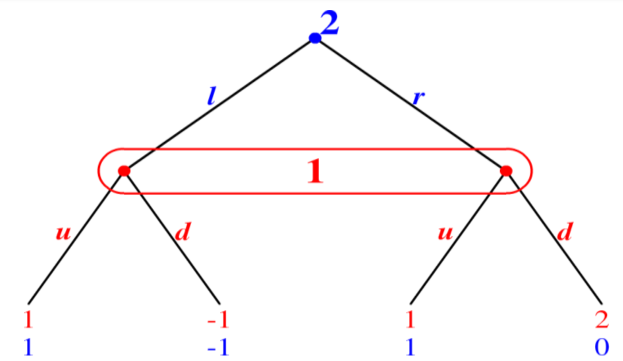




- **Cooperative or non-cooperative game:** Game is cooperative if there is an interaction between the players; it is non-cooperative if the communication is forbidden between them.
- **Others**







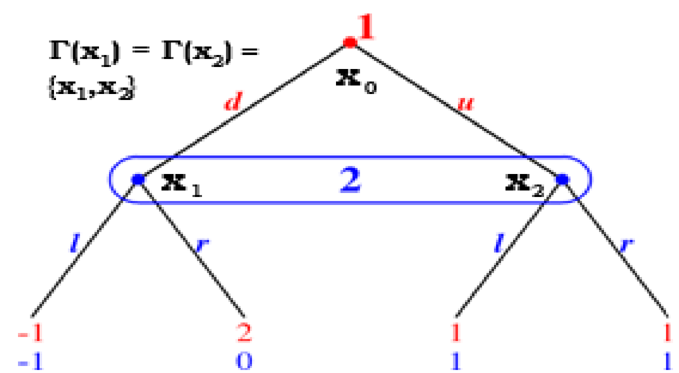
1 \ 2	*l*	*r*
d	-1, -1	2, 0
u	1, 1	1, 1

Theorem

Any extensive-form game (defined by game tree) can be assigned to a strategic-form game (defined by payoff matrix), but a strategic-form game can be assigned to several extensive-form games

Definition

Information Set: Every node x has an information set Γ(x) that partitions the nodes of the game. If x' ≠ x and if x' ∈ Γ(x), then the player who moves at x does not know whether he is at nodes x or x'.



Player 2 knows Player 1 has made move x_0 but uncertain about whether it was ***d*** or ***u***. That's why he does not know he is at nodes x_1 or x_2.

II.13. Typology by Information in Games

- All games can be classified as complete or incomplete information games.

- **Complete information games** – the player whose turn it is to move knows at least as much as those who moved before him/her.

Complete information games include:

❑ **Perfect information games** – players know the full history of the game, all moves made by all players/outcomes/payoffs, etc.

✓ The payoffs are common knowledge in the game.

✓An extensive form game without any information set.

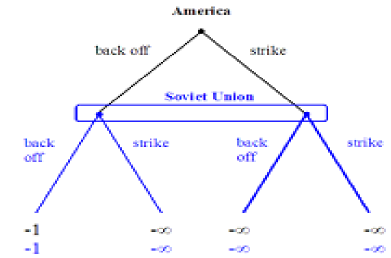
❑ **Imperfect information games** – games involving simultaneous moves where players know all possible outcomes/payoffs, but not the actions chosen by other players.

- **Incomplete information games:** At some node in the game the player whose turn it is to make a choice knows less than a player who has already moved. (player may or may not know some information about the other players, e.g., their ”type”, their strategies/preferences or payoffs.

PERFECT
COMPLETE

↑

EXTENSIVE-FORM GAMES



IMPERFECT
INCOMPLETE

↑

STRATEGIC-FORM GAMES

III. HOW TO SOLVE GAMES?

III.1. Overview

Solution Methods

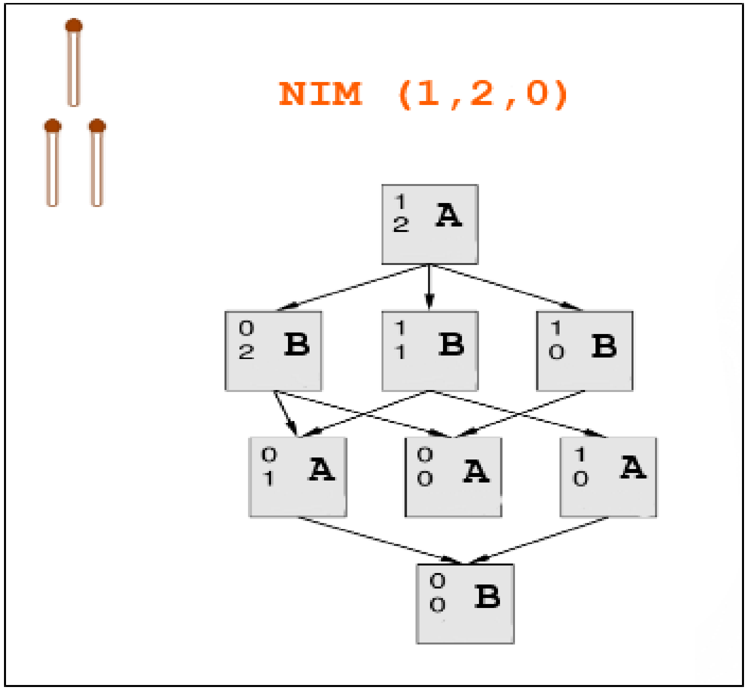
in pure strategies		**in mixed strategies**	
Sequential	**Simoultaneous**	**Zero-sum**	**Non-constant-sum**
Moves Games		**Games**	
▪ Backward Induction	▪ Minimax Method	▪ Alebraic Method	▪ Bimatrix Games:
▪ Solution in its normal form	▪ Dominated (cell-by-cell) method	▪ BRF Method	▪ Algebraic& BRF Methods in mixed

III.2. Solving a game in pure strategies

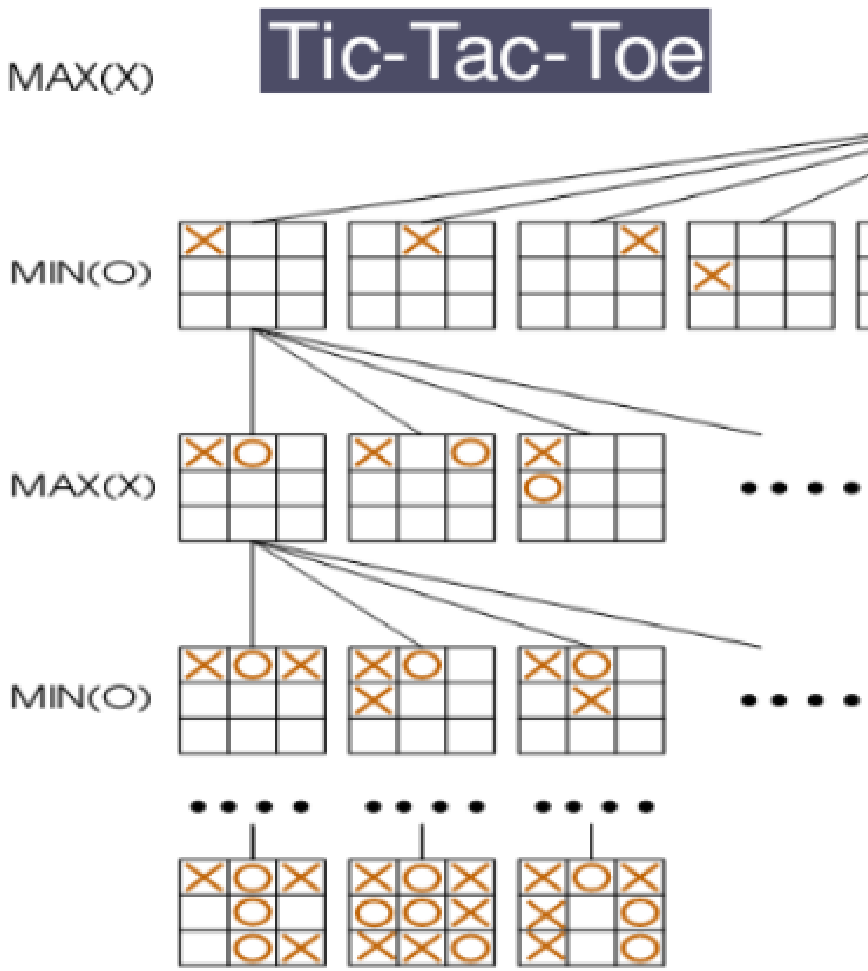
1. Sequential Games
   - Backward induction (or rollback method)
   - Solving sequential game in its normal form
2. Simultaneous Games
   - Minimax method
   - Successive elimination of dominated strategies (cell-by-cell inspection method)

- NIM as sequential game with perfect information
  - The Rule of NIM (l,m,n): there are two or three separate sets of matchstiks containing l,m and n number of sticks. The player take turn in removing one or more sticks providing they are taken from any one single set. The player who removes the last matchstick is the winner.
  - NIM is a sequential move game with perfect informationm and in NIM there is a winner strategy in general so that the first player can win if he plays well.

    Look at the the game tree NIM (1,2,0) on the right. Player 1 always win if he removes one stick from the first set of matchsticks.

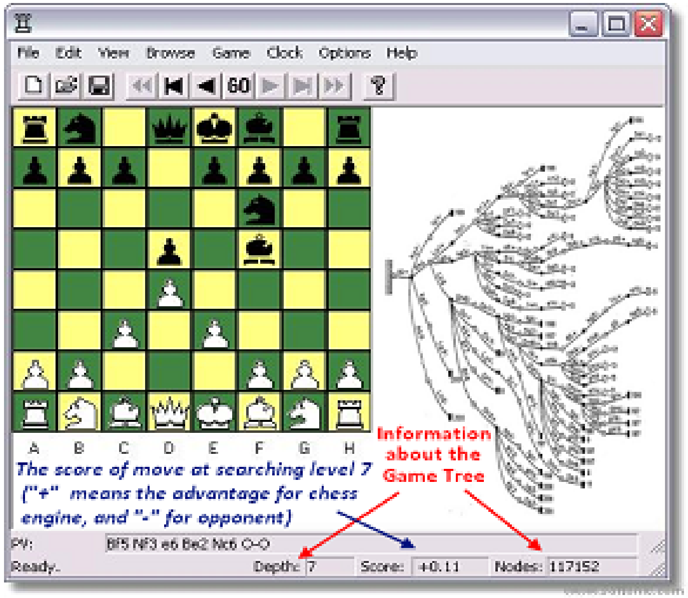


III.3. Solving a game in pure strategies



This does not work for chess because the tree is too large

- ≈ 45 and so the the deapth is 90
- ≈ 35 possible best move, thus
- $35^{90} = 10^{139}$ outcomes. However the amounts of electrons are 10^{80}

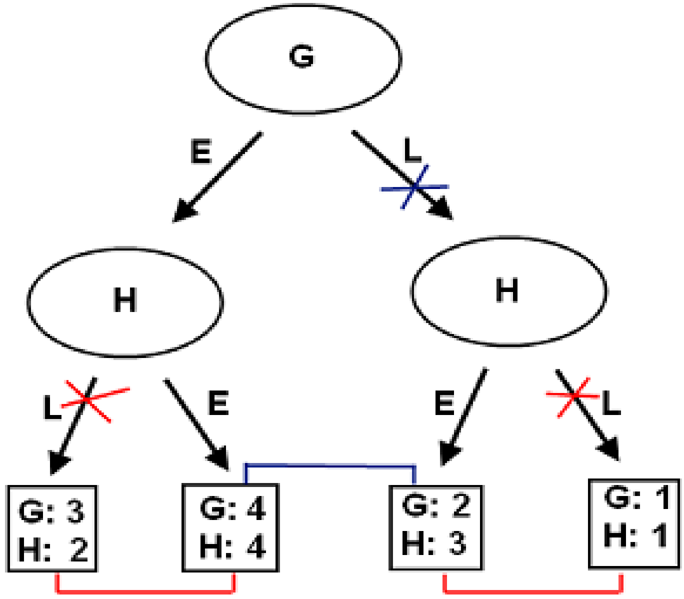


The search of a winning strategy in sequential games based on backward induction provided the game tree is constructible. If it is not, we can use **forward induction by using heuristics** (e.g. in chess computer programs). The score of the moves are calculeted by functions built in the chess program. The functions are planned by heuristics based on the theory of chess, and it can be considered as "creativity" of the engine.

- Suppose there are two players G and H, and G moves first. Start at each of the terminal nodes of the game tree. What strategy will the last player to move, player H choose starting from the immediate prior decision node of the tree?
- Compare the payoffs, player H receives at the terminal nodes, and assume player H always chooses the strategy giving him the maximal payoff.
- Continue rolling back in the same manner until you reach the root node of the tree. The indicated path is the equilibrium path. Indicate these branches of the tree.

III.5. Backward Induction

- Now treat the next-to-last node of the tree as the terminal node. Given player Hs choices, what strategy will player G choose? Again assume that player G always chooses the strategy giving him the maximal payoff. Indicate again these branches of the tree.
- Continue rolling back in the same manner until you reach the root node of the tree. The indicated path is the equilibrium path.
- **Example**



- Player H's choice:
L: 2 $\rightarrow$ E: 4 and E:3 $\leftarrow$ L: 1

- Player G's choice:
E:4 $\leftarrow$ L: 2

III.6. Equilibrium in Simultaneous Move Games

- We cannot use rollback in a simultaneous move game. So how do we find an equilibrium?
  $\mapsto$ We determine the "best response" of each player to a particular choice of strategy by the other player. We do this for both players.
  $\mapsto$ If each player's strategy choice is a best response to the strategy choice of the other player, then we have found a solution or equilibrium to the game. This solution concept is known as *the method of dominant strategies*.
- A player has a dominant strategy if it has higher payoff than all other strategies regardless of the strategies chosen by the opposing player(s).
- A game may have one or more equilibria or it might be the case it has not an equilibrium at all.
- In simple 2x2 simultaneous games we use arrows to find Nash equilibrium: if there exists common matrix elements at which arrows point in both payoff matrixes, these elements are Nash equilibrium. The arrows display the fact of domination.

III.7. The Method of Dominant Strategies

- **The Method of Dominant Strategies** . For any game including zero-sum and non-constant-sum games we can use this method.
- Take the payoff matrix of the below zero-sum game in which we can see the payoff to player G:

Player H

Player G	A	B	C
A	1	2	4
B	1	0	5
C	0	-1	-1

G eliminates row C (A, B > C) — **H eliminates column C (A, B >C)** — **G eliminates row B (A ≥ B)** here row A weakly dominates row B (1 = 1, 2 ←0)

$$\begin{pmatrix} 1 & 2 & 4 \\ 1 & 0 & 5 \\ 0 & -1 & -1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & 4 \\ 1 & 0 & 5 \\ \bullet & \bullet & \bullet \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & * \\ 1 & 0 & * \\ \bullet & \bullet & * \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & * \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{pmatrix}$$

and finally H eliminates column B: $\begin{pmatrix} 1 & * & * \\ \bullet & * & * \\ \bullet & * & * \end{pmatrix}$**. The Nash equilibrium is (A, A) = (1, -1), i.e., the payoff is 1 to G and it is -1 to H.**

III.8. The MINIMAX Method

- A zero-sum game can be solved not only by using dominant strategies but also by another method called as **Minimax**:

		Player H			
		A	B	C	MIN
Player G	A	1	2	4	1
	B	1	0	5	0
	C	0	-1	-1	-1
	MAX	1	2	5	

- Choose first the maximal elements of columns (MAX) and the minimal elements of rows (MIN), and then take the minimum of MAX elements called as the upper payoff, and the maximum of MIN elements called as the lower payoff.
  $\mapsto$ If the upper and lower payoffs are equal, there exists Nash equilibrium of the zero-sum game, and it equals to the upper and lower payoffs.
  $\mapsto$ In this game the equilibrium is (A, A) = (1, -1), exactly the same as we have received in the Slide III.5.

III.9. The MINIMAX Method in Game Tree

- Building up game tree up to a certain deapth

- Scoring the moves on the nodes

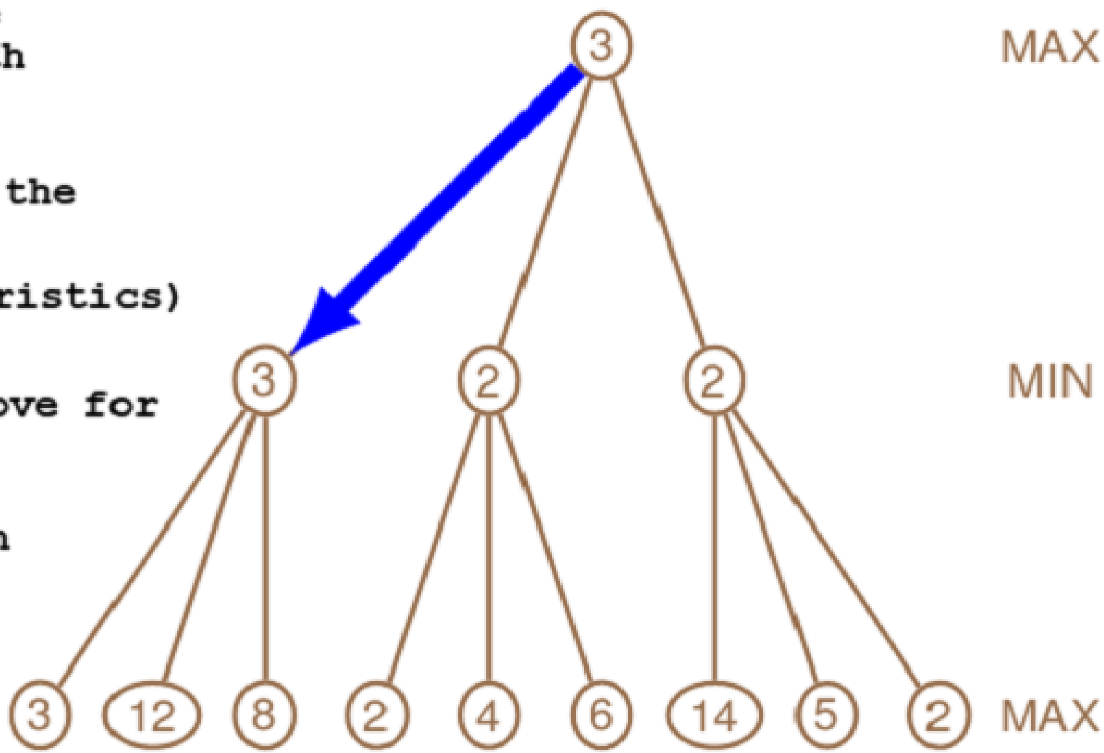
  (Value Function - Heuristics)

- Searching the best move for
  - player 1 as MAX
  - player 2 as MIN

  by Backward Induction

  or

  by Forward Induction



- To increase efficacy of MINIMAX, computer programmer use in practice other algorithms and cutting techniques.

III.10. Games with No Equilibrium and Multiple Equilibria

- A zero-sum game without equilibrium

Player B

Player A	A	B
A	2, -2	-2, 2
B	-2, 2	2, -2

M I N I M A X:

Player B

Player A	A	B	MIN
A	2	-2	-2
B	-2	2	-2
MAX	2	2	

Method of Dominant Strategies:

$$\mathbf{A:}\ \begin{pmatrix} 2 & & -2 \\ \uparrow & & \downarrow \\ -2 & & 2 \end{pmatrix} \qquad \mathbf{B:}\ \begin{pmatrix} -2 & \rightarrow & 2 \\ & & \\ 2 & \leftarrow & -2 \end{pmatrix}$$

- A bimatrix game with equilibria

Player B

Player A	A	B
A	3, 0	0, 0
B	0, 2	2, 3

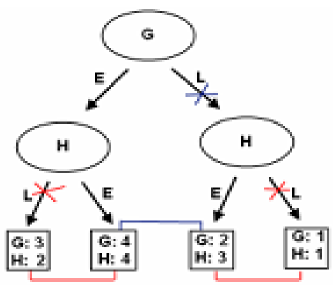
Method of Dominant Strategies:

$$\mathbf{A:}\ \begin{pmatrix} 3 & & 0 \\ \uparrow & & \downarrow \\ 0 & & 2 \end{pmatrix} \qquad \mathbf{B:}\ \begin{pmatrix} 0 & = & 0 \\ & & \\ 0 & \rightarrow & 3 \end{pmatrix}$$

Equilibria: $(A, A) = (3, 0)$ and $(B, B) = (2, 3)$

III.11. To Solve Sequential Move Game in Normal Form

- If we want to represent a sequential move game no in extensive form but in normal form, we can do it, but we need to translate the information contained in the extensive form game tree to the normal form game matrix.
- This information is the sequence of the moves: what does move the second move player H to the moves E or L of the first move player G: H replies E to E (EE) or E to L (LE), or he replies L to E (LE) or L to L (LL).



		H			
		EE	EL	LE	LL
G	E	4, 4	4, 4	3, 2	3, 2
	L	2, 3	1, 1	2, 3	1, 1
		c.1.	c.2.	c.3.	c.4.

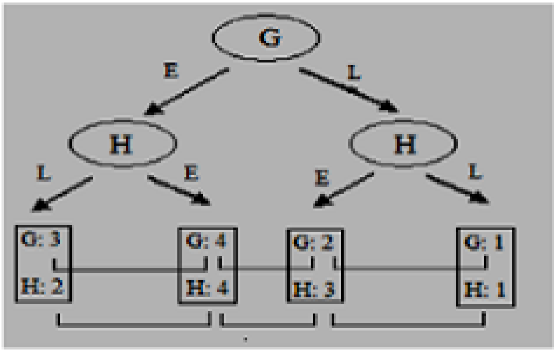
- To find the equilibrium of the game we need to verify the dominations by comparing the rows (see the arrows in the payoff matrix) and the columns of the payoff matrix:

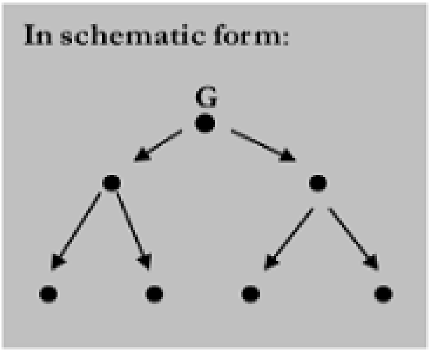
1 and 2: 4=4, 3>1; c.1 ← c.2,	3 and 4: 2=2, 3>1; c.3 ← c.4,
1 and 3: 4>2, 3=3; c.1 ← c.3,	2 and 4: 4>2, 1=1; c.2 ← c.4,
1 and 4: 4>2, 3>1; c.1 ← c.4,	2 and 3: 4>2, 1<3; c.1 = c.2.

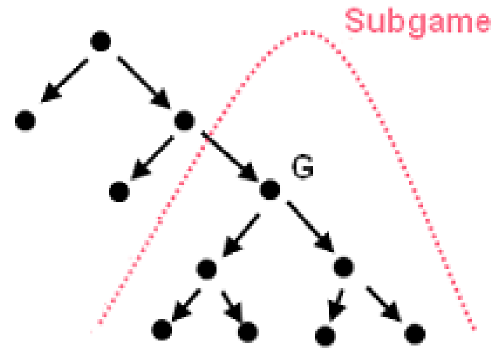
III.12. To Solve Sequential Move Game in Normal Form

- To investigate the dominations for all cells in the payoff matrix in Slide III.8., there are only two cells that are not dominated by another: (E, EE) and (E, EL). These might be candidates for equilibria.
- Why are there two equilibria in the normal form game matrix and only one (E, EE) in the extensive form game tree by the backward induction? Because (E, EL) is based on a weak and not on a strong domination.
- There is a strong domination between EE and LL ($4 > 2$ and $3 > 1$), but a weak domination between EE and EL ($4 = 4$ and $3 > 1$). So (E, EE) is the real equilibrium candidate, because (E, EL) is not the best response to the opponent's move at any node of the game tree.
- In normal form game matrix there might be additional equilibrium. Equilibrium found by backward induction to the extensive form game is referred to as Subgame Perfect Equilibrium (SPE): every player makes a perfect best response at every subgame of the tree.

- A subgame is the game that begins at any node of a game tree.



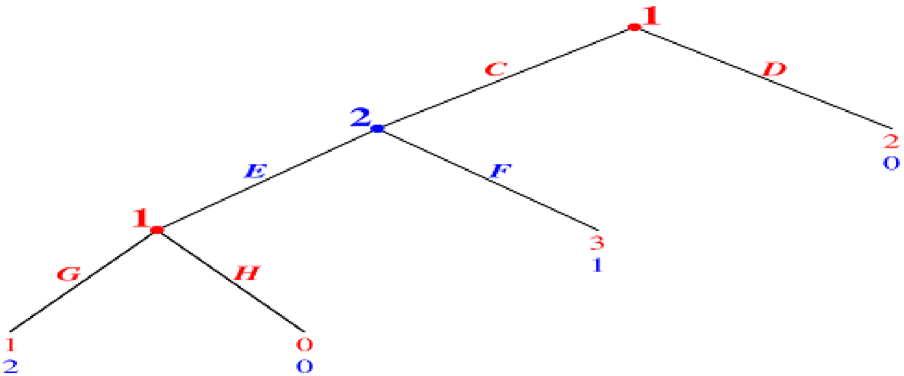




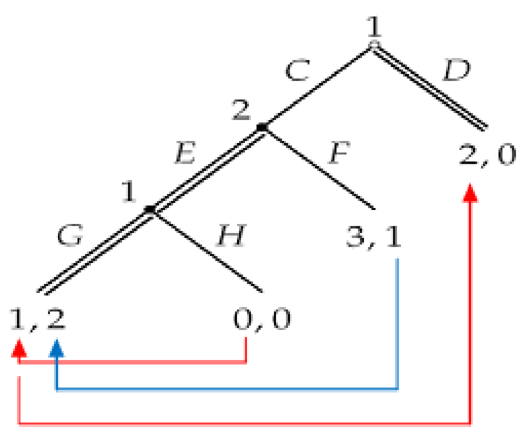
- If we apply backward induction to the larger game in which there is a subgame, the SPE of the subgame will be the equilibrium of any subgame of the complete game tree.
- **Kuhn's Theorem** : There is an SPE in any finite, perfect information game that is given in an extensive form.

III.14. SPE in Practice: an example

1 \ 2	E	F
CG	1,2	3,1
CH	0,0	3,1
DG	2,0	2,0
DH	2,0	2,0



Underlined numbers are best responses. There are 3 candidates for SPE: (DG,E), (DH,E) and (CH,F). Yet, considering the dominations in all the subgames, (DG,E) is the best responses of the two payers, and so the SPE of the whole game is (DG,E).




```
SUCCESS

Strategic form:

3 x 2 Payoff player 1

    E F
C,G 1 3
C,H 0 3
 D* 2 2

3 x 2 Payoff player 2

    E F
C,G 2 1
C,H 0 1
 D* 0 0
```

1 \ 2	E	F
C,G	1, 2	3, 1
C,H	0, 0	3, 1
D*	2, 0	2, 0

```
EE = Extreme Equilibrium, EP = Expected Payoffs

Rational:

EE 1 P1: (1) 1/2 1/2 0 EP= 3 P2: (1)   0   1 EP= 1
EE 2 P1: (2)   0   0 1 EP= 2 P2: (2) 1/2 1/2 EP= 0
EE 3 P1: (2)   0   0 1 EP= 2 P2: (3)   1   0 EP= 0
EE 4 P1: (3)   0   1 0 EP= 3 P2: (1)   0   1 EP= 1
```

mixed strategies

(D*,E) -> (DG,E) (DH,E)
(CH,F)

III.15. The Main Theorems of Game Theory

- What guarantees are there to exist Nash equilibrium and to use methods as MINIMAX to solve a game.

Theorem

Minimax theorem *: Any finite, two-player, zero-sum game has at least one Nash equilibrium in mixed strategies.*

Nash theorem *: Any finite, n-player, non-cooperative game has Nash equilibrium in mixed strategies.*

- Thus a game in mixed strategies we can find Nash equilibrium. However, to solve game in mixed strategies we need to study two solving methods: the Best Response Function and the Algebraic Method.

III.16. The Best Response Function (BRF)

A \ B	h	t
H	1 $v_B(h\|H)$ -1	-1 $v_B(t\|H)$ 1
T	-1 $v_B(h\|T)$ 1	1 $v_B(t\|T)$ -1

- Consider the "Matching pennies" game:

	A	B
Players (individuals):	A	B
Pure strategies (alternatives):	H T	h t
Mixed strategies:	q (1-q)	p (1-p)

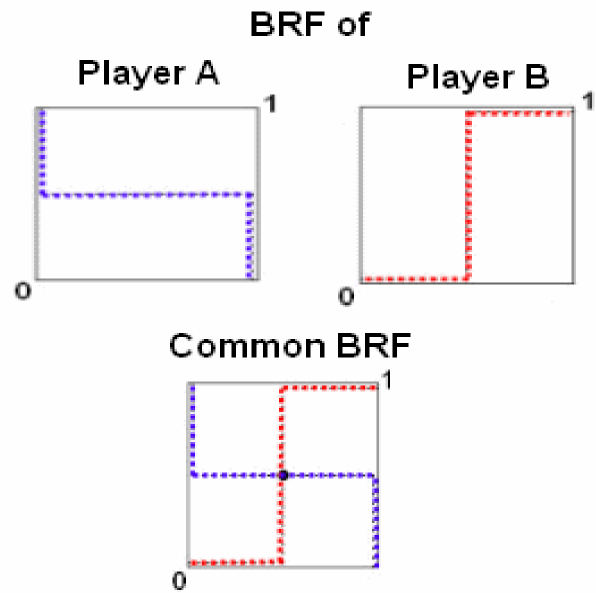
- Payoffs to Player B:

$$EU_B(h) = v_B(h|H)\,p + v_B(h|T)(1-p) = (-1)\,p + (1)(1-p)$$
$$EU_B(t) = v_B(t|H)\,p + v_B(t|T)(1-p) = (1)\,p + (-1)(1-p)$$

- It is worth for player B playing "h" if $0 \le p \le ½$. Why is the threshold 50%? Because then

$$EU_B(h) = EU_B(t) \longrightarrow v_B(h|H)\,p + v_B(h|T)(1-p) = v_B(t|H)\,p + v_B(t|T)(1-p)$$
$$\downarrow$$
$$(-1)\,p + (1)(1-p) = (1)\,p + (-1)(1-p)$$
$$\downarrow$$
$$p = ½$$

- If $½ \le p \le 1$, $EU_B(h) < EU_B(t)$, and so it is worth to play t.



- Since player B gains at the loss of player A, because the game is zero-sum, this reasoning is exactly the same for the best response of player A (q and q -1 mixed strategies), just the received mixed strategies will be inversely as to that of player B. This well appears in the Best Response Function (BRF) in the figure on the right where F corresponds to player A`s BRF and f to player B`s BRF.

- If there is an equilibrium in pure strategies we can apply minimax method to a zero-sum game. By minimax theorem, there always exists Nash equilibrium in mixed strategies even if we cannot use minimax method.
- In this case we can use the criterion of strictly determined game: if both elements of one diagonal in the payoff matrix of a 2x2 game are greater or less than the both elements of the other diagonal, we can use an algebraic method.

The mixed strategies of player 1:

$$x_1 = \frac{d-c}{(a+d)-(b+c)}$$

$$x_2 = 1 - x_1$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The mixed strategies of player 2:

$$y_1 = \frac{d-b}{(a+d)-(b+c)}$$

$$y_2 = 1 - y_1$$

- To solve the "matching pennies" game with this method, we have the same equilibrium as with BRF: both players worth playing heads and tails in fifty-fifty.

A: $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

$\alpha = (a + d) - (b + c) = 2 - (-2) = 4 > 0$

$d - c = 1 - (-1) = 2$

$x_1 = \frac{d-c}{(a+d)-(b+c)} = \frac{1}{2}$

$x_2 = 1 - x_1 = \frac{1}{2}$

B: $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$

$\beta = (a + d) - (b + c) = -2 - (-2) = -4 < 0$

$d - b = -1 - 1 = -2$

$y_1 = \frac{d-b}{(a+d)-(b+c)} = \frac{1}{2}$

$y_2 = 1 - y_1 = \frac{1}{2}$

III.18. To Solve Bimatrix Games

- Bimatrix games are 2x2 non-constant sum games. To solve them, we use a combination of BRF and algebraic method.

$$G: \begin{pmatrix} (a, a') & (b, b') \\ (c, c') & (d, d') \end{pmatrix}$$

- **Step 1 is to take the payoff matrix apart.**

$$A: \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad B: \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

- **Step 2 is to apply the algebraic method to them considering the fact that the game is non-constant sum.**

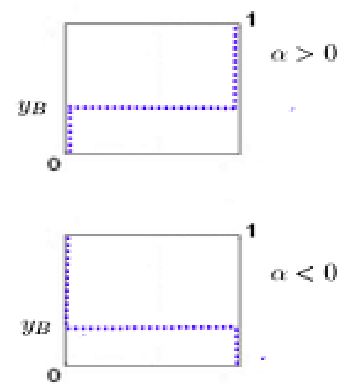
$$\alpha = (a + d) - (b + c) \qquad \beta = (a' + d') - (b' + c')$$

$$y_B = \frac{d-b}{\alpha} \qquad x_A = \frac{d'-c'}{\beta}$$

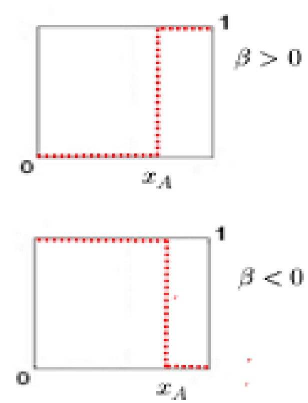
- **Step 3 is to display BRFs. Since in a bimatrix game player B does not gain necessarily at the loss of player A, thus the shape of BRF depends on the sign of α as to player A, and depends on β as to player B.**

- **Step 4 is to visualize the common BRF functions in a Cartesian system, and equilibria will be those points where BRF functions intersect each other.**

BRF of Player A



BRF of Player B



III.19. To Solve Bimatrix Games: an example

A \ B	c	d
a	4, 2	-1, -1
b	-1, -1	2, 4

Step 1

$$A: \begin{pmatrix} 4 & -1 \\ -1 & 2 \end{pmatrix} \qquad B: \begin{pmatrix} 2 & -1 \\ -1 & 4 \end{pmatrix}$$

Step 2

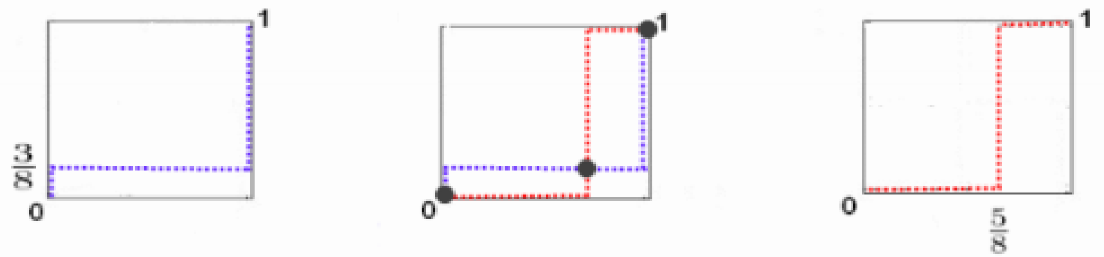
$$\alpha = (a + d) - (b + c) = 6 + 2 = 8 > 0$$

$$y_B = \frac{d-b}{\alpha} = \frac{2-(-1)}{8} = \frac{3}{8}$$

$$\beta = (a' + d') - (b' + c') = 6 + 2 = 8 > 0$$

$$x_A = \frac{d'-c'}{\beta} = \frac{4-(-1)}{8} = \frac{5}{8}$$

Step 3



Step 4

$x_1^A = 0 \quad x_1^B = 0 \qquad x_2^A = \frac{5}{8} \quad x_2^B = \frac{3}{8} \qquad x_3^A = 1 \quad x_3^B = 1$

$y_1^A = 1 \quad y_1^B = 1 \qquad y_2^A = \frac{3}{8} \quad y_2^B = \frac{5}{8} \qquad y_3^A = 0 \quad y_3^B = 0$

EQUILIBRIA: [(0,1), (0,1)] = [b, d]   [(5/8, 3/8), (3/8, 5/8)] = [mixed(a, b), mixed(c, d)]   [(1,0), (1,0)] = [a, c]